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ON q-ANALOGUES OF THE FOURIER AND HANKEL TRANSFORMS

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ABSTRACT. For H. Exton's q-analogue of the Bessel function (going back to W. Hahn in a special case, but different from F. H. Jackson's q-Bessel functions) we derive Hansen-Lommel type orthogonality relations, which, by a symmetry, turn out to be equivalent to orthogonality relations which are q-analogues of the Hankel integral transform pair. These results are implicit, in the context of quantum groups, in a paper by Vaksman and Korogodskiĭ. As a specialization we get q-cosines and q-sines which admit q-analogues of the Fourier-cosine and Fourier-sine transforms. We also get a formula which is both an analogue of Graf's addition formula and of the Weber-Schafheitlin discontinuous integral.

1. INTRODUCTION

Several possible q-analogues of the *Bessel function*

$$J_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)} \tag{1.1}$$

have been considered in the literature. The best known are two related q-Bessel functions denoted $J_\alpha^{(1)}(x;q)$ and $J_\alpha^{(2)}(x;q)$ by Ismail [10], but first introduced by Jackson in a series of papers during the years 1903–1905 (see the references in [10]) and also studied by Hahn [7]. A third q-Bessel function was introduced by Hahn [8] (in a special case; we thank G. Gasper for this reference) and by Exton [3; 4, (5.3.1.11)] (in full). In Exton's notation,

$$C_\alpha(q;x) := \frac{(1-q)^\alpha}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}\,(q^{\alpha+k+1};q)_\infty\,(-x(1-q)^2)^k}{(q;q)_k}, \tag{1.2}$$

where the q-shifted factorials are defined by

$$(a;q)_k := \prod_{j=0}^{k-1}(1-aq^j), \qquad k=1,2,\ldots; \qquad (a;q)_0 := 1; \tag{1.3}$$
$$(a;q)_\infty := \lim_{k\to\infty}(a;q)_k, \qquad |q|<1.$$

Hahn [8] considered the case $\alpha = 0$ of (1.2). They obtained these functions as the solutions of a special basic Sturm-Liouville equation, by which they could

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also derive the following q-analogue of the Fourier-Bessel orthogonality relations:

$$\int_0^1 x^\alpha C_\alpha(q;\mu_i qx)\, C_\alpha(q;\mu_j qx)\, d_qx = 0, \qquad i \neq j,\ \alpha > -1,$$

where $\mu_1, \mu_2, \ldots$ are the roots of the equation $C_\alpha(q;\mu) = 0$, and where the q-integral is defined by

$$\int_0^1 f(x)\, d_qx := (1-q) \sum_{k=0}^{\infty} f(q^k)\, q^k.$$

By specialization to $\alpha = \pm 1/2$, Exton obtained similar orthogonalities for q-analogues of the sines and cosines. So some of the harmonic analysis involving Bessel functions, sines and cosines has been extended to the q-case. However, q-analogues of the Fourier-cosine, Fourier-sine and Hankel transforms were missing until now (except for a q-Laplace transform with inversion formula, cf. Hahn [7, §9] and Feinsilver [5]).

Exton [4, (5.3.3.1)] also generalized the generating function

$$e^{z(t-t^{-1})/2} = \sum_{k=-\infty}^{\infty} t^k J_k(z) \tag{1.4}$$

(cf. Watson [15, §2.1(1)]) to the case of his q-Bessel functions (1.2):

$$e_q((1-q)t)\, E_q(-(1-q)t^{-1}x) = \sum_{n=-\infty}^{\infty} t^n C_n(q;x), \tag{1.5}$$

where

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1, \tag{1.6}$$

and

$$E_q(z) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{(q;q)_k} = (-z;q)_\infty \tag{1.7}$$

are q-analogues of the exponential function (cf. Gasper and Rahman [6, (1.3.15), (1.3.16)]).

Recently, Vaksman and Korogodskiĭ [14] gave an interpretation of the q-Bessel functions (1.2) as matrix elements of irreducible representations of the quantum group of plane motions. Their paper, which does not contain proofs, implicitly contains some new orthogonality relations for the functions (1.2). In particular, the unitariness of the representations implies a q-analogue of the Hansen-Lommel orthogonality relations

$$\delta_{nm} = \sum_{k=-\infty}^{\infty} J_{k+n}(x)\, J_{k+m}(x), \qquad n, m \in \mathbb{Z} \tag{1.8}$$

(cf. [15, §2.5(3),(4)]). Furthermore, the Schur type orthogonality relations for matrix elements of irreducible unitary representations which are square integrable with respect to a suitable Haar functional yield a q-analogue of Hankel's Fourier-Bessel integral

$$f(x)=\int_0^\infty J_\alpha(xt)\left(\int_0^\infty J_\alpha(ty)f(y)y\,dy\right)t\,dt \tag{1.9}$$

(cf. [15, §14.3]).

It is the purpose of the present note to state these two types of orthogonality relations for the functions (1.2) explicitly, to show that the first type is immediately implied by the generating function (1.5), and to rewrite the first type as the second type by use of a simple, but possibly new, symmetry for the functions (1.2). This will be done in §2. In §3 we will show that the second type of orthogonality is, on the one hand, a limit case of the orthogonality for the little q-Jacobi polynomials and, on the other hand, allows the Hankel transform inversion formula as a limit case for $q \uparrow 1$. In §4 we will generalize the two orthogonality relations to two equivalent formulas, which are respectively the q-analogues of Graf's addition formula and the Weber-Schafheitlin discontinuous integral. The special cases of the q-Fourier-cosine and sine transforms will be the topic of §5. No material from §4 is needed in this section. Finally, Appendix A will contain rigorous proofs of two limit results.

In a recent paper by Rahman [13], where a q-analogue of the Fourier-Bessel orthogonality for Jackson's q-Bessel functions is discussed, the author states in his concluding remarks that Jackson's q-Bessel functions probably have nicer properties than those of Exton. However, the results obtained in [14] and in the present paper might suggest that the Hahn-Exton functions are more suitable for harmonic analysis, both within and without the context of quantum groups. Future research will help to clarify the merits of the various types of q-Bessel functions.

In this paper we will not preserve Exton's notation $C_\alpha(q\,;x)$ in (1.2), but state the results in terms of the q-hypergeometric function

$${}_1\phi_1(0\,;w\,;q\,,z):=\sum_{k=0}^\infty \frac{(-1)^k q^{k(k-1)/2} z^k}{(w\,;q)_k\,(q\,;q)_k}\,. \tag{1.10}$$

Our motivation is that (i) the ${}_r\phi_s$ notation is fairly well known nowadays and clarifies the position of these q-Bessel functions among other q-hypergeometric functions; (ii) q-analysis should not depend too much on the $q=1$ case, so scaling factors simplifying the limit transition $q\uparrow 1$ easy should not be hidden in the definitions of q-special functions; and (iii) the classical definition (1.1) of Bessel functions is very natural in the context of the generating function (1.4) and for analysis concentrating on J_0, but it is less fortunate when the focus is on another Bessel function of fixed order (cf. $J_{-1/2}(x)$ versus $\cos x$), so one should be very careful before fixing the definition and notation of a q-Bessel function. Therefore, the notations for q-Bessel functions, q-cosines and q-sines in §§3 and 5 should be considered as ad hoc notations, which are only used locally in this paper for clarifying the analogy of the functions with the $q=1$ case.

2. Symmetry and orthogonality for q-Bessel functions

We will always assume that $0 < q < 1$. The general q-hypergeometric series is defined by

(2.1)

$$ {}_r\phi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k ((-1)^k q^{k(k-1)/2})^{s-r+1} z^k}{(b_1, \ldots, b_s; q)_k (q; q)_k}, $$

where the q-shifted factorial is defined by (1.3) and

$$ (a_1, \ldots, a_r; q)_k := (a_1; q)_k (a_2; q)_k \cdots (a_r; q)_k. $$

The upper and lower parameters in the left-hand side of (2.1) may also be written on one line as $a_1, \ldots, a_r; b_1, \ldots, b_s$. The power series, in the nonterminating case of (2.1), has radius of convergence ∞, 1 or 0 according to whether $r - s < 1$, $= 1$, or > 1, respectively (see [6, Chapter 1] for further details). Thus the defining formula (1.2) for the Hahn-Exton q-Bessel function can be rewritten as

$$ C_\alpha(q; x) := \frac{(1-q)^\alpha (q^{\alpha+1}; q)_\infty}{(q; q)_\infty} \, {}_1\phi_1(0; q^{\alpha+1}; q, x(1-q)^2) $$

and the q-exponential functions (1.6), (1.7) can be written as ${}_1\phi_0(0; -; q, z)$ and ${}_0\phi_0(-; -; q, -z)$, respectively.

Our object will be the Hahn-Exton q-Bessel function written as the q-hypergeometric series (1.10). It is well defined for $z, w \in \mathbb{C}$ with w outside $\{1, q^{-1}, q^{-2}, \ldots\}$. These singularities can be removed by multiplication by $(w; q)_\infty$:

$$ (w; q)_\infty \, {}_1\phi_1(0; w; q, z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (q^k w; q)_\infty z^k}{(q; q)_k}. \tag{2.2} $$

Proposition 2.1. *The series in* (2.2) *defines an entire analytic function in* z, w, *which is also symmetric in* z, w:

$$ (w; q)_\infty \, {}_1\phi_1(0; w; q, z) = (z; q)_\infty \, {}_1\phi_1(0; z; q, w). \tag{2.3} $$

Both sides can be majorized by

$$ (-|z|; q)_\infty (-|w|; q)_\infty. \tag{2.4} $$

Proof. Substitute for $(q^k w; q)_\infty$ in (2.2) the ${}_0\phi_0$ series given by (1.7):

$$ (w; q)_\infty \, {}_1\phi_1(0; w; q, z) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} q^{kl} \frac{(-1)^l q^{l(l-1)/2} w^l}{(q; q)_l} \frac{(-1)^k q^{k(k-1)/2} z^k}{(q; q)_k}. $$

The summand of the double series can be majorized by

$$ \frac{q^{l(l-1)/2} |w|^l}{(q; q)_l} \frac{q^{k(k-1)/2} |z|^k}{(q; q)_k}. $$

Thus the double sum converges absolutely, uniformly for z, w in compacta, and it is symmetric in z and w. □

Remark 2.2. Formula (2.3) is a limit case of Heine's transformation formula

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} \, {}_2\phi_1\left[\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right] \tag{2.5} $$

(cf. [6, (1.4.1)]). Indeed, first let $b \to 0$, then replace z by z/a and let $a \to \infty$. At least formally, by termwise limits, we obtain (2.3).

Remark 2.3. For $w := q^{1-n}$ $(n = 1, 2, \dots)$ we interpret the left hand side of (2.2) by the series at its right-hand side. Then the first n terms vanish, so the summation starts with $k = n$. When we make the change of summation variable $k = n + l$, we obtain

$$\begin{aligned}(q^{1-n}; q)_\infty \, {}_1\phi_1(0; q^{1-n}; q, z) \\ = (-1)^n q^{n(n-1)/2} z^n (q^{n+1}; q)_\infty \, {}_1\phi_1(0; q^{n+1}; q, q^n z)\end{aligned} \tag{2.6}$$

for $n \in \mathbb{Z}$. (The case $n < 0$ follows from the case $n > 0$ of (2.6) by changing z into $q^{-n} z$.)

Remark 2.4. Because of (2.6), the behaviour of the two equal sides of (2.3) as $|w| \to \infty$ (cf. (2.4)) drastically improves when w runs over the values q^{1-n}, $n = 1, 2, \dots$. For such w we can majorize these expressions by

$$q^{n(n-1)/2} |z|^n (-|z|; q)_\infty (-q; q)_\infty .$$

We will now restate Exton's generating function (1.5) in terms of the notation (2.2), and also give the short proof, for reasons of completeness.

Proposition 2.5. *For* $z, t \in \mathbb{C}$ *such that* $0 < |t| < |z|^{-1}$ *there is the absolutely convergent expansion*

$$e_q(tz) E_q(-t^{-1} z) = \frac{(t^{-1} z; q)_\infty}{(tz; q)_\infty} = \sum_{n=-\infty}^{\infty} t^n z^n \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \, {}_1\phi_1(0; q^{n+1}; q, z^2) \tag{2.7}$$

$$= \sum_{n=-\infty}^{\infty} t^n z^n \frac{(z^2; q)_\infty}{(q; q)_\infty} \, {}_1\phi_1(0; z^2; q, q^{n+1}). \tag{2.8}$$

Proof. Expansion of the left-hand side of (2.7) gives

$$\begin{aligned}&\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} t^{l-k} z^{l+k}}{(q; q)_k (q; q)_l} \\ &= \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(q^{l+1}; q)_\infty (-1)^k q^{k(k-1)/2} t^{l-k} z^{l+k}}{(q; q)_\infty (q; q)_k},\end{aligned}$$

which is an absolutely convergent double sum for $z, t \in \mathbb{C}$ such that $0 < |t| < |z|^{-1}$. Now pass to new summation variables k, n by substituting $l = k + n$. This yields, by substitution of (2.2), the right-hand side of (2.7). Formula (2.8) follows by the symmetry (2.3). □

In §3 we will show that (2.8) is a q-analogue of an infinite integral of Weber and Sonine. Replace t by t^{-1} in (2.7) and multiply the new identity by the original identity. The resulting formula

$$\begin{aligned}1 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} t^{n-m} z^n \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \, {}_1\phi_1(0; q^{n+1}; q, z^2) \\ \times z^m \frac{(q^{m+1}; q)_\infty}{(q; q)_\infty} \, {}_1\phi_1(0; q^{m+1}; q, z^2)\end{aligned} \tag{2.9}$$

is absolutely convergent for $z, t \in \mathbb{C}$ such that $t \neq 0$ and $|z| < |t| < |z|^{-1}$. So equality of coefficients of equal powers of t at both sides yields a q-analogue of the orthogonality relations (1.8):

Proposition 2.6. *For* $|z| < 1$ *and* $n, m \in \mathbb{Z}$ *we have*

$$\sum_{k=-\infty}^{\infty} z^{k+n} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{n+k+1}; q, z^2) \times z^{k+m} \frac{(q^{m+k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{m+k+1}; q, z^2) = \delta_{nm} \tag{2.10}$$

and

$$\sum_{k=-\infty}^{\infty} z^{k+n} \frac{(z^2; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; z^2; q, q^{n+k+1}) \times z^{k+m} \frac{(z^2; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; z^2; q, q^{m+k+1}) = \delta_{nm}, \tag{2.11}$$

where the sums on the left-hand sides are absolutely convergent, uniformly on compact subsets of the open unit disk.

Formula (2.11) follows from (2.10) by the symmetry (2.3). In §3 we will show that (2.11) is a q-version of Hankel's Fourier-Bessel integral (1.9).

Remark 2.7. Analogous to Proposition 2.5 there are the two generating functions for the q-Bessel functions $J_\alpha^{(1)}(x; q)$ and $J_\alpha^{(2)}(x; q)$ of Jackson and Ismail [10]:

$$\begin{aligned} e_q(tz)\, e_q(-t^{-1}z) &= \frac{1}{(tz; q)_\infty (-t^{-1}z; q)_\infty} \\ &= \sum_{n=-\infty}^{\infty} t^n z^n \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} {}_2\phi_1(0, 0; q^{n+1}; q, -z^2), \end{aligned} \tag{2.12}$$

for $|z| < |t| < |z|^{-1}$, and

$$\begin{aligned} E_q(tz)\, E_q(-t^{-1}z) &= (-tz; q)_\infty (t^{-1}z; q)_\infty \\ &= \sum_{n=-\infty}^{\infty} t^n z^n \frac{q^{n(n-1)/2} (q^{n+1}; q)_\infty}{(q; q)_\infty} {}_0\phi_1(-; q^{n+1}; q, -q^n z^2). \end{aligned} \tag{2.13}$$

In a way similar to (2.10) we can now derive the biorthogonality relations

$$\sum_{k=-\infty}^{\infty} z^{k+n} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_2\phi_1(0, 0; q^{n+k+1}; q, -z^2) \times z^{k+m} q^{(m+k)(m+k-1)/2} \frac{(q^{m+k+1}; q)_\infty}{(q; q)_\infty} \times {}_0\phi_1(-; q^{m+k+1}; q, -q^{m+k} z^2) = \delta_{nm}, \tag{2.14}$$

valid for $|z| < 1$ and $n, m \in \mathbb{Z}$. The case $n = m$ of this result goes back to Jackson (see also [7, §8]). Formula (2.14) can be rewritten in several ways by substitution of the transformations

$${}_0\phi_1(-; c; q, cz) = (z; q)_\infty\, {}_2\phi_1(0, 0; c; q, z) = \frac{1}{(c; q)_\infty} {}_1\phi_1(z; 0; q, c).$$

However, this will not transform (2.14) into an orthogonality; it remains a biorthogonality.

3. Some limit transitions

Jacobi polynomials tend to Bessel functions:

$$\frac{P_{n_N}^{(\alpha,\beta)}(1-x^2/(2N^2))}{P_{n_N}^{(\alpha,\beta)}(1)} = {}_2F_1\left(-n_N\,,\, n_N+\alpha+\beta+1\,;\,\alpha+1\,;\,\frac{x^2}{4N^2}\right)$$
$$\xrightarrow{N\to\infty} {}_0F_1\left(-\,;\,\alpha+1\,;\,-\left(\frac{\lambda x}{2}\right)^2\right) = \left(\frac{\lambda x}{2}\right)^{-\alpha}\Gamma(\alpha+1)\,J_\alpha(\lambda x),$$

where n_N/N tends to λ for $N\to\infty$. When this limit transition is applied to the formula which recovers a function from its Fourier-Jacobi coefficients, we obtain, at least formally, Hankel's Fourier-Bessel integral (1.9).

The q-analogue of this limit transition starts with the *little q-Jacobi polynomials*

$$p_n(x\,;\,a\,,\,b\,;\,q) := {}_2\phi_1(q^{-n}\,,\,abq^{n+1}\,;\,aq\,;\,q\,,\,qx),$$

which satisfy orthogonality relations

$$\tag{3.1} \begin{aligned} &\frac{(qa\,,\,qb\,;\,q)_\infty}{(q\,,\,q^2ab\,;\,q)_\infty}\sum_{k=0}^{\infty}(p_np_m)(q^k\,;\,a\,,\,b\,;\,q)\,(qa)^k\,\frac{(q^{k+1}\,;\,q)_\infty}{(q^{k+1}b\,;\,q)_\infty} \\ &\quad= \frac{(qa)^n\,(1-qab)\,(qb\,,\,q\,;\,q)_n}{(1-q^{2n+1}ab)\,(qa\,,\,qab\,;\,q)_n}\,\delta_{nm}, \end{aligned}$$

where $0<a<q^{-1}$, $b<q^{-1}$ (see Andrews and Askey [2]).

It is clear that

$$p_{N-n}(q^Nx\,;\,a\,,\,b\,;\,q) = {}_2\phi_1(q^{-N+n}\,,\,abq^{N-n+1}\,;\,aq\,;\,q\,,\,q^{N+1}x)$$

tends formally (termwise) to ${}_1\phi_1(0\,;\,aq\,;\,q\,,\,q^{n+1}x)$ as $N\to\infty$. (See Proposition A.1 for a rigorous proof of this limit result.) Also, when we replace n, m, k in (3.1) by $N-n$, $N-m$, $N+k$, respectively (so the sum runs from $-N$ to ∞), and when we let $N\to\infty$, we obtain as a formal (termwise) limit the orthogonality relations (2.11).

In order to see that (2.11) is a q-analogue of Hankel's Fourier-Bessel integral (1.9), rewrite (2.11) as the transform pair

$$\tag{3.2} \begin{cases} g(q^n) = \displaystyle\sum_{k=-\infty}^{\infty} q^{(k+n)(\alpha+1)}\,\frac{(q^{2\alpha+2}\,;\,q^2)_\infty}{(q^2\,;\,q^2)_\infty}\,{}_1\phi_1(0\,;\,q^{2\alpha+2}\,;\,q^2\,,\,q^{2k+2n+2})\,f(q^k), \\ f(q^k) = \displaystyle\sum_{n=-\infty}^{\infty} q^{(k+n)(\alpha+1)}\,\frac{(q^{2\alpha+2}\,;\,q^2)_\infty}{(q^2\,;\,q^2)_\infty}\,{}_1\phi_1(0\,;\,q^{2\alpha+2}\,;\,q^2\,,\,q^{2k+2n+2})\,g(q^n), \end{cases}$$

where f, g are L^2-functions on the set $\{q^k \mid k\in\mathbb{Z}\}$ with respect to counting measure. With the ad hoc notation

$$\tag{3.3} J_\alpha(z\,;\,q^2) := \frac{z^\alpha\,(q^{2\alpha+2}\,;\,q^2)_\infty}{(q^2\,;\,q^2)_\infty}\,{}_1\phi_1(0\,;\,q^{2\alpha+2}\,;\,q^2\,,\,q^2z^2)$$

(different from Rahman's proposal in [13, (1.13)]) and with the replacement of $f(q^k)$, $g(q^n)$ by $q^k f(q^k)$, $q^n g(q^n)$ this becomes

$$g(q^n) = \sum_{k=-\infty}^{\infty} q^{2k} J_\alpha(q^{k+n}; q^2) f(q^k),$$

(3.4)

$$f(q^k) = \sum_{n=-\infty}^{\infty} q^{2n} J_\alpha(q^{k+n}; q^2) g(q^n).$$

Now observe that

$$J_\alpha((1-q)z; q^2) = \frac{(1+q)^{-\alpha} z^\alpha}{\Gamma_{q^2}(\alpha+1)} \, {}_1\phi_1\left(0; q^{2\alpha+2}; q^2, (1-q^2)^2 \left(\frac{qz}{1+q}\right)^2\right)$$

converges, for $q \uparrow 1$, to the Bessel function

$$J_\alpha(z) = \frac{2^{-\alpha} z^\alpha}{\Gamma(\alpha+1)} \, {}_0F_1\left(-; \alpha+1; \frac{-z^2}{4}\right),$$

where we used Proposition A.2 and the fact that

$$\frac{(1-q^2)^\alpha (q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{1}{\Gamma_{q^2}(\alpha+1)} \to \frac{1}{\Gamma(\alpha+1)}$$

as $q \uparrow 1$ (cf. Andrews [1, Appendix A], Koornwinder [11, Appendix B]). We can apply this to (3.4) when we let $q \uparrow 1$ under the side condition that $\frac{\log(1-q)}{\log q} \in 2\mathbb{Z}$. For such q we can replace q^k, q^n in (3.4) by $(1-q)^{1/2} q^k$, $(1-q)^{1/2} q^n$, and next $f((1-q)^{1/2} q^k)$, $g((1-q)^{1/2} q^n)$ by $f(q^k)$, $g(q^n)$. With the q-integral notation

$$\int_0^\infty h(z)\, d_q z := (1-q) \sum_{j=-\infty}^{\infty} h(q^j) q^j,$$

(3.4) then takes the form

$$\begin{cases} g(\lambda) = \displaystyle\int_0^\infty f(x)\, J_\alpha((1-q)\lambda x; q^2) x \, d_q x, \\ f(x) = \displaystyle\int_0^\infty g(\lambda)\, J_\alpha((1-q)\lambda x; q^2) \lambda \, d_q \lambda, \end{cases}$$

where λ in the first identity and x in the second identity take the values q^n, $n \in \mathbb{Z}$. For $q \uparrow 1$ we obtain, at least formally, the Hankel transform pair

$$\begin{cases} g(\lambda) = \displaystyle\int_0^\infty f(x)\, J_\alpha(\lambda x) x \, dx, \\ f(x) = \displaystyle\int_0^\infty g(\lambda)\, J_\alpha(\lambda x) \lambda \, d\lambda, \end{cases}$$

which is equivalent to (1.9).

In order to find the classical formula corresponding to (2.8), we rewrite (2.8) first in terms of the notation (3.3):

$$\frac{(q^{\alpha+1-t}; q^2)_\infty}{(q^{\alpha+1+t}; q^2)_\infty} = \sum_{n=-\infty}^{\infty} q^n q^{nt} J_\alpha(q^n; q^2), \qquad \Re t > -\Re\alpha - 1.$$

For $\log(1-q)/\log q \in \mathbb{Z}$ this can be rewritten as

$$\frac{(1+q)^t \Gamma_{q^2}((\alpha+1+t)/2)}{\Gamma_{q^2}((\alpha+1-t)/2)} = \int_0^\infty x^t J_\alpha((1-q)x\,;q^2)\, d_q x.$$

Formally, as $q \uparrow 1$, this yields

$$\frac{2^t \Gamma((\alpha+1+t)/2)}{\Gamma((\alpha+1-t)/2)} = \int_0^\infty x^t J_\alpha(x)\, dx\,,$$

which formula, valid for $-1/2 > \Re t > -\Re\alpha - 1$, goes back to Weber and Sonine, (cf. [15, 13.24(1)]).

4. A q-ANALOGUE OF GRAF'S ADDITION FORMULA

In this section we will generalize the considerations which led to the orthogonality relations in Proposition 2.6. The resulting formula will turn out to be a q-analogue of Graf's addition formula and, at the same time, of the discontinuous integral of Weber and Schafheitlin. In a final remark we will point out that the Graf type addition formula part can also be done for the q-Bessel functions of Jackson and Ismail, by similar methods, which go back to Heine, the founding father of q-hypergeometric series.

A formula more general than (2.9) can be derived by expanding the expression

$$\frac{(xs^{-1}t\,;q)_\infty\,(yt^{-1}\,;q)_\infty}{(yt\,;q)_\infty\,(xst^{-1}\,;q)_\infty} \tag{4.1}$$

as a Laurent series in t $(|sx| < |t| < |y|^{-1})$ in two different ways. On the one hand, (4.1) can be expanded by twofold substitution of the q-binomial formula (cf. [6, (1.3.2)]) as

$$\begin{aligned}
&{}_1\phi_0(s^{-1}xy^{-1}\,;-\,;q\,,yt)\,{}_1\phi_0(s^{-1}yx^{-1}\,;-\,;q\,,xst^{-1})\\
&\quad=\sum_{k=0}^\infty\sum_{l=-\infty}^\infty \frac{(s^{-1}yx^{-1}\,;q)_k\,(s^{-1}xy^{-1}\,,q^{l+1}\,;q)_\infty}{(q\,;q)_k\,(q^l s^{-1}xy^{-1}\,,q\,;q)_\infty} s^k x^k y^l t^{l-k}\\
&\quad=\sum_{n=-\infty}^\infty t^n y^n \sum_{k=0}^\infty \frac{(s^{-1}yx^{-1}\,;q)_k\,(s^{-1}xy^{-1}\,,q^{n+k+1}\,;q)_\infty}{(q\,;q)_k\,(q^{n+k}s^{-1}xy^{-1}\,,q\,;q)_\infty}(sxy)^k\,,
\end{aligned}$$

where we substituted $l = k+n$. Since the inner sum in the last part can be expressed in terms of a ${}_2\phi_1$ series, we obtain the identity

$$\begin{aligned}
&\frac{(xs^{-1}t\,;q)_\infty\,(yt^{-1}\,;q)_\infty}{(yt\,;q)_\infty\,(xst^{-1}\,;q)_\infty}\\
&\quad=\sum_{n=-\infty}^\infty t^n y^n \frac{(s^{-1}xy^{-1}\,,q^{n+1}\,;q)_\infty}{(q^n s^{-1}xy^{-1}\,,q\,;q)_\infty}\\
&\qquad\times {}_2\phi_1\left[\begin{matrix} q^n s^{-1}xy^{-1}\,,s^{-1}yx^{-1} \\ q^{n+1}\end{matrix}\,;q\,,sxy\right], \qquad |sx| < |t| < |y|^{-1}.
\end{aligned} \tag{4.2}$$

Here we use, for $n < 0$, an interpretation of the ${}_2\phi_1$ similar to our convention in Remark 2.3. The case that $s^{-1}xy^{-1}$ is an integer power of q is then understood by continuity in s, x, y. The analogue of (2.6) becomes

(4.3)

$$\frac{(s^{-1}xy^{-1}, q^{1+n}; q)_\infty}{(q^n s^{-1}xy^{-1}, q; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^n s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{1+n} \end{matrix} ; q, sxy \right]$$
$$= (sxy)^{-n} \frac{(s^{-1}yx^{-1}, q^{1-n}; q)_\infty}{(q^{-n}s^{-1}yx^{-1}, q; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^{-n}s^{-1}yx^{-1}, s^{-1}xy^{-1} \\ q^{1-n} \end{matrix} ; q, sxy \right].$$

Note that (4.2) reduces to (2.7) in the special case $x = 0$ (and also, in view of (2.6), for $y = 0$). Formula (4.2) is a q-analogue of (1.4) with z and t replaced by $2(y - s^{-1}x)^{1/2}(y - sx)^{1/2}$ and $t(y - s^{-1}x)^{1/2}(y - sx)^{-1/2}$, respectively.

On the other hand we expand (4.1) by twofold substitution of (2.7):

$$\begin{aligned} &\frac{(xs^{-1}t; q)_\infty (yt^{-1}; q)_\infty}{(yt; q)_\infty (xst^{-1}; q)_\infty} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} t^n s^k y^{n+k} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{n+k+1}; q, y^2) \\ &\qquad \times x^k \frac{(q^{k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{k+1}; q, x^2), \end{aligned} \tag{4.4}$$

which generalizes (2.9). When we compare coefficients of equal powers of t in (4.2) and (4.4), we obtain

Proposition 4.1. *For* $|sxy| < 1$ *we have*

$$\begin{aligned} &y^n \frac{(s^{-1}xy^{-1}, q^{n+1}; q)_\infty}{(q^n s^{-1}xy^{-1}, q; q)_\infty} {}_2\phi_1 \left[\begin{matrix} q^n s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{n+1} \end{matrix} ; q, sxy \right] \\ &= \sum_{k=-\infty}^{\infty} s^k y^{n+k} \frac{(q^{n+k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{n+k+1}; q, y^2) \\ &\qquad \times x^k \frac{(q^{k+1}; q)_\infty}{(q; q)_\infty} {}_1\phi_1(0; q^{k+1}; q, x^2). \end{aligned} \tag{4.5}$$

Formula (4.5) is a q-analogue of the addition formula

$$\left(\frac{y - s^{-1}x}{y - sx} \right)^{n/2} J_n(\sqrt{(y - s^{-1}x)(y - sx)}) = \sum_{k=-\infty}^{\infty} s^k J_{n+k}(y) J_k(x), \tag{4.6}$$

due to Graf (cf. [15, §11.3 (1)]). The special case $n = 0$ of (4.5) is a q-analogue of Neumann's addition formula for Bessel functions J_0 (cf. [15, §11.2 (1)]). In the special case $x = y$, $s = 1$ the left-hand side of (4.5) becomes $y^n \delta_{n,0}$, so then (4.5) reduces to the orthogonality relations (2.10).

When we apply the symmetries (2.5) and (4.3) to the left-hand side and (2.3) to the right-hand side of (4.5), and replace n by $n - m$ and then k by $k + m$, we obtain an equivalent identity:

Proposition 4.2. *For* $|sxy| < 1$ *we have*

(4.7)
$$\begin{aligned} s^{-m}\, y^{n-m}\, &\frac{(s^{-1}xy^{-1},\, y^2;\, q)_\infty}{(sxy,\, q;\, q)_\infty}\, {}_2\phi_1\left[\begin{matrix} qsx^{-1}y,\, sxy \\ y^2 \end{matrix};\, q,\, q^{n-m}s^{-1}xy^{-1}\right] \\ &= s^{-n}\, x^{m-n}\, \frac{(s^{-1}yx^{-1},\, x^2;\, q)_\infty}{(sxy,\, q;\, q)_\infty}\, {}_2\phi_1\left[\begin{matrix} qsxy^{-1},\, sxy \\ x^2 \end{matrix};\, q,\, q^{m-n}s^{-1}yx^{-1}\right] \\ &= \sum_{k=-\infty}^{\infty} s^k\, y^{n+k}\, \frac{(y^2;\, q)_\infty}{(q;\, q)_\infty}\, {}_1\phi_1(0;\, y^2;\, q,\, q^{n+k+1}) \\ &\qquad \times x^{m+k}\, \frac{(x^2;\, q)_\infty}{(q;\, q)_\infty}\, {}_1\phi_1(0;\, x^2;\, q,\, q^{m+k+1}). \end{aligned}$$

In particular, for $x = y$ we have

(4.8)
$$\begin{aligned} s^{-m}\, z^{n-m}\, &\frac{(s^{-1},\, z^2;\, q)_\infty}{(sz^2,\, q;\, q)_\infty}\, {}_2\phi_1\left[\begin{matrix} qs,\, sz^2 \\ z^2 \end{matrix};\, q,\, q^{n-m}s^{-1}\right] \\ &= s^{-n}\, z^{m-n}\, \frac{(s^{-1},\, z^2;\, q)_\infty}{(sz^2,\, q;\, q)_\infty}\, {}_2\phi_1\left[\begin{matrix} qs,\, sz^2 \\ z^2 \end{matrix};\, q,\, q^{m-n}s^{-1}\right] \\ &= \sum_{k=-\infty}^{\infty} s^k\, z^{n+k}\, \frac{(z^2;\, q)_\infty}{(q;\, q)_\infty}\, {}_1\phi_1(0;\, z^2;\, q,\, q^{n+k+1}) \\ &\qquad \times z^{m+k}\, \frac{(z^2;\, q)_\infty}{(q;\, q)_\infty}\, {}_1\phi_1(0;\, z^2;\, q,\, q^{m+k+1}), \end{aligned}$$

where $|sz^2| < 1$. This can be considered as a kind of Poisson kernel for the orthogonal system with orthogonality relations (2.11). Inspection of the first two parts of (4.8) shows that this kernel is positive if $0 < z < 1$ and $1 < s < \min\{q^{-1},\, z^{-2}\}$.

Let us look for the classical analogue of formula (4.7). In (4.7) first replace q by q^2, then x, y, s by $q^{\alpha+1}, q^{\beta+1}, q^{-\gamma-1}$, respectively. Then, with the notation (3.3), formula (4.7) can be rewritten as

$$\begin{aligned} \sum_{k=-\infty}^{\infty} &q^{k(-\gamma+1)}\, J_\alpha(q^{m+k};\, q^2)\, J_\beta(q^{n+k};\, q^2) \\ &= \begin{cases} q^{n\beta}\, q^{m(\gamma-\beta-1)}\, \dfrac{(q^{\alpha-\beta+\gamma+1},\, q^{2\beta+2};\, q^2)_\infty}{(q^{\alpha+\beta-\gamma+1},\, q^2;\, q^2)_\infty} \\ \quad \times {}_2\phi_1(q^{\beta-\alpha-\gamma+1},\, q^{\alpha+\beta-\gamma+1};\, q^{2\beta+2};\, q^2,\, q^{2n-2m+\alpha-\beta+\gamma+1}), \\ q^{m\alpha}\, q^{n(\gamma-\alpha-1)}\, \dfrac{(q^{\beta-\alpha+\gamma+1},\, q^{2\alpha+2};\, q^2)_\infty}{(q^{\alpha+\beta-\gamma+1},\, q^2;\, q^2)_\infty} \\ \quad \times {}_2\phi_1(q^{\alpha-\beta-\gamma+1},\, q^{\alpha+\beta-\gamma+1};\, q^{2\alpha+2};\, q^2,\, q^{2m-2n+\beta-\alpha+\gamma+1}), \end{cases} \end{aligned}$$

where $\Re(\alpha + \beta - \gamma + 1) > 0$. Now replace q^k by $q^k(1-q)$ (with $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$) and let m, n depend on q such that, as $q \uparrow 1$, q^m tends to a and q^n tends to b. Depending on whether $b < a$ or $b > a$, make the formal limit transition $q \uparrow 1$ in the first or second identity, respectively. Then, for $\Re(\alpha+\beta-\gamma+1) > 0$,

$\Re\gamma > -1$, we obtain the discontinuous integral of Weber and Schafheitlin:

$$2^{\gamma}\int_0^{\infty} x^{-\gamma} J_{\alpha}(ax) J_{\beta}(bx)\, dx$$
$$= \begin{cases} \dfrac{a^{\gamma-\beta-1}\, b^{\beta}\, \Gamma((\alpha+\beta-\gamma+1)/2)}{\Gamma((\alpha-\beta+\gamma+1)/2)\Gamma(\beta+1)} \\ \quad \times {}_2F_1\left(\dfrac{\beta-\alpha-\gamma+1}{2}, \dfrac{\alpha+\beta-\gamma+1}{2}; \beta+1; \dfrac{b^2}{a^2}\right) & \text{if } b < a, \\ \dfrac{a^{\alpha}\, b^{\gamma-\alpha-1}\, \Gamma((\alpha+\beta-\gamma+1)/2)}{\Gamma((\beta-\alpha+\gamma+1)/2)\Gamma(\alpha+1)} \\ \quad \times {}_2F_1\left(\dfrac{\alpha-\beta-\gamma+1}{2}, \dfrac{\alpha+\beta-\gamma+1}{2}; \alpha+1; \dfrac{a^2}{b^2}\right) & \text{if } a < b, \end{cases}$$

(cf. [15, §13.4 (2)]). Note that the two analytic expressions the right-hand side are no longer equal, as they were in the q-case.

Remark 4.3. Analogous to (4.2) we have

$$\begin{aligned} \frac{(xs^{-1}t; q)_{\infty}(-xst^{-1}; q)_{\infty}}{(yt; q)_{\infty}(-yt^{-1}; q)_{\infty}} &= \sum_{n=-\infty}^{\infty} t^n y^n \frac{(s^{-1}xy^{-1}, q^{n+1}; q)_{\infty}}{(q^n s^{-1}xy^{-1}, q; q)_{\infty}} \\ &\quad \times {}_2\phi_1\left[\begin{matrix} q^n s^{-1}xy^{-1}, sxy^{-1} \\ q^{n+1} \end{matrix}; q, -y^2\right], \end{aligned} \tag{4.9}$$

for $|y| < |t| < |y|^{-1}$. The case $s = 1$ of (4.9) goes back to Heine [9, p. 121] (see also Hahn [7, §8]). Its special cases $(x, y, s) = (0, z, 1)$ and $(-z, 0, 1)$ are the formulas (2.12) and (2.13). Like (4.2), formula (4.9) is a q-analogue of (1.4) with z and t replaced by $2(y-s^{-1}x)^{1/2}(y-sx)^{1/2}$ and $t(y-s^{-1}x)^{1/2} \times (y-sx)^{-1/2}$, respectively. Similarly to (4.5) we obtain from (2.12), (2.13), and (4.9) that, for $|y| < 1$:

$$\begin{aligned} & y^n \frac{(s^{-1}xy^{-1}, q^{n+1}; q)_{\infty}}{(q^n s^{-1}xy^{-1}, q; q)_{\infty}}\, {}_2\phi_1\left[\begin{matrix} q^n s^{-1}xy^{-1}, sxy^{-1} \\ q^{n+1} \end{matrix}; q, -y^2\right] \\ &= \sum_{k=-\infty}^{\infty} s^k y^{k+n} \frac{(q^{k+n+1}; q)_{\infty}}{(q; q)_{\infty}}\, {}_2\phi_1(0, 0; q^{k+n+1}; q, -y^2) \\ &\quad \times x^k q^{k(k-1)/2} \frac{(q^{k+1}; q)_{\infty}}{(q; q)_{\infty}}\, {}_0\phi_1(-; q^{k+1}; q, -q^k x^2). \end{aligned} \tag{4.10}$$

The special case $n = 0$ of this formula is the limit case $\nu \downarrow 0$ of Rahman's addition formula [12, (1.10)]. Like (4.5), formula (4.10) is a q-analogue of Graf's addition formula (4.6). The special case $x = y$, $s = 1$ of (4.10) gives the biorthogonality relations (2.14).

5. q-ANALOGUES OF THE FOURIER-COSINE AND FOURIER-SINE TRANSFORM

Put

$$\cos(z;q^2) := {}_1\phi_1(0;q;q^2,q^2z^2)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k}}{(q;q)_{2k}} \tag{5.1}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{-1/2}(z;q^2) \tag{5.2}$$

and

$$\sin(z;q^2) := (1-q)^{-1} z\, {}_1\phi_1(0;q^3;q^2,q^2z^2)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} z^{2k+1}}{(q;q)_{2k+1}} \tag{5.3}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{1/2}(z;q^2). \tag{5.4}$$

Here we have used the notation (3.3). (The functions introduced above should not be confused with the functions $\cos_q$ and $\sin_q$ considered in [6, Exercise 1.14].) Clearly we have the formal (termwise) limits

$$\cos((1-q)z;q^2) \to \cos z \quad \text{and} \quad \sin((1-q)z;q^2) \to \sin z$$

as $q \uparrow 1$. By Proposition A.2 these limit transitions hold pointwise, uniformly on compacta. When we substitute (5.2) or (5.4) in (3.4) and replace $f(q^k)$, $g(q^n)$ by $q^{-k/2} f(q^k)$, $q^{-n/2} g(q^n)$, we obtain the transform pairs

$$\begin{cases} g(q^n) = \dfrac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \displaystyle\sum_{k=-\infty}^{\infty} q^k \begin{Bmatrix} \cos(q^{k+n};q^2) \\ \text{or} \\ \sin(q^{k+n};q^2) \end{Bmatrix} f(q^k), \\ f(q^k) = \dfrac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \displaystyle\sum_{n=-\infty}^{\infty} q^n \begin{Bmatrix} \cos(q^{k+n};q^2) \\ \text{or} \\ \sin(q^{k+n};q^2) \end{Bmatrix} g(q^n). \end{cases} \tag{5.5}$$

The transformations $f \mapsto g$ and $g \mapsto f$ of (5.5) establish an isometry of Hilbert spaces:

$$\sum_{k=-\infty}^{\infty} q^k |f(q^k)|^2 = \sum_{n=-\infty}^{\infty} q^n |g(q^n)|^2.$$

Now let $q \uparrow 1$ under the side condition that $\frac{\log(1-q)}{\log q} \in 2\mathbb{Z}$. Replace q^k, q^n in (5.5) by $(1-q)^{1/2}q^k$, $(1-q)^{1/2}q^n$, and then $f((1-q)^{1/2}q^k)$, $g((1-q)^{1/2}q^n)$

by $f(q^k)$, $g(q^n)$. Then (5.5) takes the form

$$\begin{cases} g(\lambda) = \dfrac{(1+q)^{1/2}}{\Gamma_{q^2}(1/2)} \displaystyle\int_0^\infty f(x) \begin{Bmatrix} \cos((1-q)\lambda x\,;\,q^2) \\ \text{or} \\ \sin((1-q)\lambda x\,;\,q^2) \end{Bmatrix} d_q x\,, \\ f(x) = \dfrac{(1+q)^{1/2}}{\Gamma_{q^2}(1/2)} \displaystyle\int_0^\infty g(\lambda) \begin{Bmatrix} \cos((1-q)\lambda x\,;\,q^2) \\ \text{or} \\ \sin((1-q)\lambda x\,;\,q^2) \end{Bmatrix} d_q \lambda. \end{cases}$$

Formally, as $q \uparrow 1$, we obtain the classical Fourier pairs

$$g(\lambda) = \sqrt{2/\pi} \int_0^\infty f(x)\cos(\lambda x)\,dx\,, \qquad f(x) = \sqrt{2/\pi} \int_0^\infty g(\lambda)\cos(\lambda x)\,d\lambda$$

and

$$g(\lambda) = \sqrt{2/\pi} \int_0^\infty f(x)\sin(\lambda x)\,dx\,, \qquad f(x) = \sqrt{2/\pi} \int_0^\infty g(\lambda)\sin(\lambda x)\,d\lambda.$$

With the notation

$$(D_q f)(z) = D_{q,z} f(z) := \frac{f(z) - f(qz)}{(1-q)z}$$

for the q-derivative, we obtain from (5.1) and (5.3) that

$$(1-q)D_{q,z} \cos(z\,;\,q^2) = -q \sin(qz\,;\,q^2)\,,$$
$$(1-q)D_{q,z} \sin(z\,;\,q^2) = \cos(z\,;\,q^2).$$

Hence

$$(1-q)^2 (D_q^2 f)(q^{-1}z) = \begin{cases} -q^2\lambda^2 f(z) & \text{if } f(z) = \cos(\lambda z\,;\,q^2), \\ -q\lambda^2 f(z) & \text{if } f(z) = \sin(\lambda z\,;\,q^2). \end{cases}$$

So the two systems of functions $z \mapsto \cos(q^n z)$ $(n \in \mathbb{Z})$ and $z \mapsto \sin(q^n z)$ $(n \in \mathbb{Z})$ have disjoint eigenvalues with respect to the operator which sends f to the function

$$z \mapsto (1-q)^2 (D_q^2 f)(q^{-1}z) = qz^{-2}(qf(q^{-1}z) - (1+q)f(z) + f(qz)).$$

This operator also has the selfadjointness property

$$\sum_{k=-\infty}^{\infty} q^k (D_q^2 f)(q^{k-1}) g(q^k) = \sum_{k=-\infty}^{\infty} q^k f(q^k)(D_q^2 g)(q^{k-1})$$

for f, g of finite support on $\{q^k \mid k \in \mathbb{Z}\}$.

Observe that the q-deformation of d^2/dx^2 considered above yields a symmetry breaking. The two-dimensional eigenspaces of d^2/dx^2 are broken apart into one-dimensional eigenspaces. Therefore it does not seem to be very useful to consider a q-exponential built from the functions defined by (5.1) and (5.3). Any linear combination $f(z)$ of $\cos(\lambda z\,;\,q^2)$ and $\sin(\lambda z\,;\,q^2)$ will no longer satisfy an eigenfunction equation

$$(1-q)^2\,(D_q^2 f)(q^{-1}z) = \mu\, f(z), \tag{5.6}$$

while the nice function

$$\begin{aligned} f(z) &:= {}_1\phi_1(0\,;\,-q^{1/2}\,;\,q^{1/2}\,,\,\pm iq^{3/4}\lambda z) \\ &= \cos(\lambda z\,;\,q^2) \mp iq^{1/4}\sin(q^{1/2}\lambda z\,;\,q^2) \end{aligned}$$

(cf. Exton [4, 5.2.2.1]), which satisfies (5.6) with $\mu = -q^2\lambda^2$, no longer remains within the spectral decompositions implied by (5.5).

Appendix A. Rigorous proofs of some limit results

In this appendix we will give proofs of the limit transitions from little q-Jacobi polynomials to q-Bessel functions and from q-Bessel functions to ordinary Bessel functions.

Proposition A.1. *For* $0 < a < q^{-1}$ *and* $0 \le b < q^{-1}$ *we have*

$$\lim_{n\to\infty} {}_2\phi_1(q^{-n}\,,\,q^{n+1}ab\,;\,qa\,;\,q\,,\,q^n x) = {}_1\phi_1(0\,;\,qa\,;\,q\,,\,x),$$

uniformly for x *in compact subsets of* $\mathbb{C}$.

Proof. Put

$$\begin{aligned} R_n(x) &:= {}_2\phi_1(q^{-n}\,,\,q^{n+1}ab\,;\,qa\,;\,q\,,\,q^n x) - {}_1\phi_1(0\,;\,qa\,;\,q\,,\,x) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k\, q^{k(k-1)/2}\, x^k}{(qa\,;\,q)_k\,(q\,;\,q)_k} \left(-1 + \prod_{j=1}^{k}(1-q^{n-j+1})\,(1-q^{n+j}ab)\right). \end{aligned}$$

Since

$$\prod_{j=1}^{k}(1-x_j) \ge 1 - \sum_{j=1}^{k} x_j \quad \text{if } 0 \le x_j \le 1\,,\ j = 1\,,\ldots,k\,,$$

we have

$$\begin{aligned} &\left|-1 + \prod_{j=1}^{k}(1-q^{n-j+1})\,(1-q^{n+j}ab)\right| \\ &\qquad \le \sum_{j=1}^{k} q^{n-j+1} + \sum_{j=1}^{k} q^{n+j}ab = \frac{q^{n+1}}{1-q}\,((1-q^k)ab - 1 + q^{-k}). \end{aligned}$$

Thus, for $|x| \le M$,

$$|R_n(x)| \le \frac{q^{n+1}}{1-q} \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2}\, M^k}{(qa\,;\,q)_k\,(q\,;\,q)_k}\,((1-q^k)ab - 1 + q^{-k}),$$

where the infinite sum converges for all $M > 0$ by d'Alembert's ratio test. □

Proposition A.2. *For* $\alpha > -1$ *we have*

$$\lim_{q\uparrow 1} {}_1\phi_1(0\,;\,q^{\alpha+1}\,;\,q\,,\,(1-q)^2 z) = {}_0F_1(-\,;\,\alpha+1\,;\,-z),$$

uniformly for z *in compact subsets of* $\mathbb{C}$.

Proof.

$$ {}_1\phi_1(0; q^{\alpha+1}; q, (1-q)^2 z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (1-q)^{2k} z^k}{(q^{\alpha+1}; q)_k (q; q)_k} $$

and the summand in the sum on the right-hand side can be majorized by

$$ (q^{-\alpha/2}|z|)^k \prod_{j=0}^{k-1} \frac{(q^{(\alpha+j)/2} - q^{1+(\alpha+j)/2})(q^{j/2} - q^{1+j/2})}{(1-q^{1+\alpha+j})(1-q^{1+j})}. \tag{A.1} $$

Now, by [11, Lemma A.1] (read $-1 \leq \mu - \lambda$ instead of $0 \leq \mu - \lambda$ in the formulation of that lemma), we see that

$$ \frac{q^{(\alpha+j)/2} - q^{1+(\alpha+j)/2}}{1 - q^{1+\alpha+j}} $$

increases to $(1+\alpha+j)^{-1}$ as $q \uparrow 1$ if $\alpha + j \geq 0$. So, the expression in (A.1) for $1/2 < q < 1$ is dominated by

$$ \frac{(2^{\alpha/2}|z|)^k}{(\alpha+1)_k\, k!} \quad \text{if } \alpha \geq 0 $$

and by

$$ \text{const} \frac{|z|^k}{(\alpha+1)_k\, k!} \quad \text{if } -1 < \alpha < 0. $$

So the proposition follows by dominated convergence. □

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